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REMARKS ON CLOSED MINIMAL SUBMANIFOLDS IN THE STANDARD RIEMANNIAN m-SPHERE

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Introduction

An r-dimensional minimal submanifold N^r of a Riemannian manifold M^m is a regularly imbedded sub-Riemannian manifold, which locally gives an extremal for r-dimensional volume, with fixed boundary, and therefore is the r-dimensional generalization of a geodesic curve [3, §52]. For this type of global geometric variational problem, it is naturally interesting to find the *closed* r-dimensional minimal submanifolds of a given Riemannian manifold M^m; in this respect, very few such submanifolds are known even in the simplest and nicest case in which M^m is the standard sphere S^m. Intuitively, closed minimal submanifolds of codimension one should be "fewer" than those with higher codimension, and hence "harder" to get. So far, all the known examples of closed minimal submanifolds of codimension one in S^m are "homogeneous" (see [6] for definition) and they are the extremal orbits of suitable isometry subgroups respectively.

In this short note, we shall begin with an observation that every *homogeneous* minimal submanifold N^r in S^m is *algebraic* in the following sense:

Consider S^m as the unit sphere in the euclidean space $\boldsymbol{R}^{m+1}$. Let $N^r \subseteq S^m$ be a closed minimal submanifold in S^m, O the origin of $\boldsymbol{R}^{m+1}$ and $\overrightarrow{Ox}$ the ray from O passing through a point $x \in \boldsymbol{R}^{m+1}$. Then it is clear that the cone

$$ON^r = \text{the union of } \overrightarrow{Ox} \text{ with } x \in N^r$$

is also *minimal in the euclidean space* $\boldsymbol{R}^{m+1}$ [1, §§6.15, 10.2]. We shall call N^r real algebraic if ON^r is a real algebraic cone.

It is well known that every complex algebraic cone is always minimal in $\boldsymbol{C}^n = \boldsymbol{R}^{2n}$ [4, §§4.2, 4.10]. However, the codimension of a complex algebraic cone is at least two. Hence the problem of *real algebraic minimal cones of codimension one* seems to be more delicate than the complex case. Another motivation to investigate closed minimal subminifolds of codimension one in the standard sphere S^m is due to their closed relationship with the problem of interior regularity for non-parametric solutions to Plateau's problem in codimension 1 and higher dimensional generalizations of Bernstein's theorem [2].

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In §1, we give a complete list of *one-codimensional homogeneous* minimal submanifolds in the standard spheres by using the representation theory of compact Lie groups. We also derive a simple algebraic condition for the equation of an algebraic cone to be minimal, and then use the condition to show the following interesting coincidence:

The classically known examples of closed minimal submanifolds of condimension one in S^m, namely, those of the type S^{m-1} and $S^p \times S^q$, $(p+q = m-1;\ p, q \geq 1)$, are *exactly* the *real algebraic minimal submanifolds of degree* 1 *and* 2.

In addition to the homogeneous examples given in the list of observation 2, §1, we produce some new algebraic minimal submanifolds which are no longer homogeneous. Hence the set of algebraic minimal submanifolds in S^m is essentially larger than that of homogeneous ones. It is then quite interesting to classify real algebraic minimal submanifolds of degree higher than two up to equivalence under the orthogonal transformations. It turns out that the algebraic difficulties involved in such a problem are rather formidable.

As a by-product, we derive the existence of some kind of a normal form for homogeneous polynomials of arbitrary degree and arbitrary number of variables over *real closed fields with respect to the orthogonal linear substitutions,* and also the following theorem, which is an analogy of a theorem of Frankel [5], on the geometry of one-codimensional closed minimal submanifolds of compact homogeneous Riemannian manifolds.

Theorem. *Let $M^m = G/H$ be a compact connected manifold with homogeneous Riemannian metric, and N_1, N_2 two closed minimal submanifolds of codimension one in M^m. Then either $N_1 \cap N_2 \neq \phi$ or N_1, N_2 are geodesically parallel.*

1. Observations

Observation 1. *Every homogeneous minimal submanifold of S^m is real algebraic.*

A minimal submanifold N^r in a Riemannian manifold M^m is called homogeneous if N^r is also an orbit of a *suitable isometry subgroup*, $G \subseteq ISO(M^m)$, of M^m. It was proved in [6, p. 5] that a submanifold $N^r \subseteq M^m$ is homogeneous minimal if and only if it is an *extremal orbit* of a suitable isometry subgroup $G \subseteq ISO(M^m)$. In the particular case in which $M^m = S^m$ and we are interested, G acts orthogonally on R^{m+1}. It is well known in the classical invariant theory that there exists a finite basis, say $\{P_i(x_1, \cdots, x_{m+1}),\ i = 1, 2, \cdots, s\}$, of the ring of invariant polynomials, $\mathfrak{D}(G)$, with respect to G. It is clear that invariant functions of G may be naturally considered as functions of the orbit space R^{m+1}/G, In the case where G is a *compact linear group,* we have the following lemma:

Lemma 1. *If G is a compact linear group operating on R^{m+1}, then the set* *of fundamental invariant homogeneous polynomials* $\{P_i(x),\ i = 1, 2, \cdots, s\}$ *separates points as functions of the orbit space* R^{m+1}/G.

Proof. Suppose A and B are two different orbits of G in R^{m+1}, Since G is compact, A, B are compact and there exists a continuous function f of R^{m+1} with constant values 0 and 1 on A and B respectively. We may approximate f over a large compact set containing A, B by a polynomial function $\tilde{f}$. Let

$$f_1 = \int_G (\sigma \cdot \tilde{f}) d\sigma$$

where $d\sigma$ is the normalized Haar measure of G. Then f_1 is an invariant polynomial such that $f_1(A) \neq f_2(B)$. But it is well known that any invariant polynomial can be expressed integrally in terms of the fundamental invariants $P_i(x)$. Hence, we must have $P_i(A) \neq P_i(B)$ for at least one i, namely, the set of functions $\{P_i(x)\}$ separates points on the orbit space R^{m+1}/G.

Observation 1 follows readily from Lemma 1.

Observation 2. By using the representation theory of compact Lie groups and some computations in the invariance theory, one gets the following list of *one-codimensional* homogeneous minimal submanifolds of the standard sphere S^m and the degree of the algebraic equations of their corresponding cones.

G	ϕ	m	N^{m-1}	degree of the equation of $ON^{(m-1)}$
$SO(m)$	$\rho_m \oplus \theta$	m	S^{m-1}	1
$SO(p+1) \times SO(q+1)$	$\rho_{p+1} \oplus \rho_{q+1}$	$p+q+1$	$S^p \times S^q$	2
$SO(3)$	$\sigma^2\rho_3 - \theta$	4	$SO(3)/(z_2 + z_2)$	3
$SU(3)$	$Ad_{SU(3)}$	7	$SU(3)/T^2$	3
$Sp(2)$	$Ad_{Sp(2)}$	9	$Sp(2)/T^2$	4
$SO(n) \times SO(2)$	$\rho_n \otimes \rho_2$	$2n-1$	$\dfrac{SO(n) \times SO(2)}{SO(n-2) \times z_2}$	4
$SU(n) \times SU(2)$	$\mu_n \otimes_C \mu_2$	$4n-1$	$\dfrac{SU(n) \times SU(2)}{SU(n-2) \times T^1}$	4
$Sp(n) \times Sp(2)$	$\nu_{2n} \otimes_Q \nu_4$	$8n-1$	$\dfrac{Sp(n) \times Sp(2)}{Sp(n-2) \times Sp(1)^2}$	4
G_2	Ad_{G_2}	13	G_2/T^2	6
F_4	ϕ_1	25	$F_4/\text{Spin}(8)$	

where ρ_n, μ_n and ν_{2n} are the natural representations of $SO(n)$, $SU(n)$ and $Sp(n)$ on R^n, C^n and $Q^n = C^{2n}$ respectively. As an example, let us work out the third case in detail.

Let $\mathfrak{S}_3$ be the vector space of quadratic forms of 3 real variables, and $\mathfrak{S}_3'$ the subspace of quardratic forms with trace zero. It is clear that $SO(3)$ acts naturally on $\mathfrak{S}_3$ as *linear substitutions* and $\mathfrak{S}_3'$ is invariant with respect to the action of $SO(3)$. As usual, one may represent $\mathfrak{S}_3$ as the set of 3×3 symmetric matrices, and the above $SO(3)$ action is then given by conjugation. It is a well known fact that every symmetric matrix can be transformed into a diagonal matrix by orthogonal conjugation, and the coefficients of its characteristic polynomial form a complete set of invariants. By looking at the stability subgroups of those diagonal matrices, it is quite easy to see that the principal orbit type is $SO(3)/(z_2 + z_2)$ and of codimension one in the unit sphere, S^4, of $\mathfrak{S}_3'$. Further analysis of the orbit structure will show that the *largest principal orbit A* in $S^4 \subseteq \mathfrak{S}_3'$ is characterized by $\operatorname{tr}(A) = 0$ and $\det(A) = 0$.

Observation 3. Let $F(x_0, x_1, \cdots, x_m)$ be a homogeneous irreducible polynomial such that $F(x_0, x_1, \cdots, x_m) = 0$ defines an m-dimensional cone $\mathscr{C}$ in R^{m+1}. Then $\mathscr{C}$ is minimal if and only if

$$(1) \qquad \left(\sum_{i=0}^{m} \frac{\partial^2 F}{\partial x_i^2}\right) \cdot \left(\sum_{i=0}^{m} \left(\frac{\partial F}{\partial x_i}\right)^2\right) - \sum_{i,j=0}^{m} \frac{\partial^2 F}{\partial x_i \partial x_j} \cdot \frac{\partial F}{\partial x_i} \cdot \frac{\partial F}{\partial x_j} = 0 \quad (\text{mod } F).$$

Proof. Let ∇F be the gradient vector field on $\mathscr{C}$, and X, Y two tangent vectors of $\mathscr{C}$ at a regular point p on $\mathscr{C}$. Then the so-called second fundamental bilinear form of $\mathscr{C}$ at p is given by

$$B_p(X, Y) = \frac{1}{|\nabla F|} \langle D_X \cdot \nabla F, Y \rangle$$

where $D_X \cdot \nabla F$ is the directional derivative of the gradient vector field with respect to X. Hence, $\mathscr{C}$ is minimal if and only if its mean curvature

$$(2) \qquad m(p) = \text{trace } B_p(X, Y) = 0$$

for all regular points $p \in \mathscr{C}$. It is not difficult to show that (2) is equivalent to (1). If we denote the Laplacian and Hessian operators by Δ and H respectively, then (1) may be abbreviated as

$$(1') \qquad (\Delta F) \cdot |\nabla F|^2 - \nabla F \cdot HF \cdot \nabla F^t \equiv 0 \quad (\text{mod } F).$$

Observation 4. Naturally, we shall first look at real algebraic minimal cones of lower degrees. It is obvious that every linear homogeneous equation is minimal, and that all linear homogeneous equations are congruent to each other under the group of motions, and their intersections with the unit sphere S^m are those "equators." If one tries to classify minimal quadratic cones under the group of motions, one finds that there is exactly one minimal quadratic cone $Q(p, q)$ in R^{m+1} for each non-ordered integer pair $p, q > 0$ and $p + q = m - 1$. The equation of $Q(p, q)$ can be put into the following normal form

$$q\cdot(x_0^2 + x_1^2 + \cdots + x_p^2) - p(x_{p+1}^2 + \cdots + x_m^2) = 0 . \tag{3}$$

The intersections of $Q(p, q)$ with the unit sphere S^m are exactly the classically known examples of minimal hypersurfaces in S^m of the form $S^p \times S^q$, $p + q = m - 1$. Hence, it is quite interesting to notice that those classical examples are just the real algebraic examples of degrees one and two.

Proof. Suppose $Q(x)$ is a minimal quadratic. One may assume that $Q(x)$ is already in the normal form, namely, $Q(x) = \sum_{i=0}^{m} a_i x_i^2 (a_0 \geq a_1 \geq \cdots \geq a_p > 0 > a_{p+1} \geq \cdots \geq a_m)$. It follows from equation (1′) that

$$\Delta Q \cdot |\nabla Q|^2 - \nabla Q \cdot HQ \cdot \nabla Q^t = k \cdot Q$$

for a suitable constant k. That is,

$$\left(2 \sum_{i=0}^{m} a_i\right) \cdot \left(\sum_{i=0}^{m} (2a_i x_i)^2\right) - \sum_{i=0}^{m} 2a_i (2a_i x_i)^2 = k \cdot \sum_{i=0}^{m} a_i x_i^2$$

or

$$\left(\sum_{i=0}^{m} a_i\right) \cdot a_j - a_j^2 = k' \qquad (j = 0, 1, \cdots, m) .$$

Direct computation shows that

$$a_0 = a_1 = \cdots a_p \text{ and } a_{p+1} = \cdots = a_m ; \; a_0/a_m = - q/p .$$

On the other hand, it is easy to see that the equation for the examples $S^p \times S^q \subseteq S^m$ given in the list may be written as (3). Notice that the set $S^p \times S^q \subseteq S^m \subseteq R^{m+1} = R^{p+1} \times R^{q+1}$ is given as follows:

$(\mathfrak{X}, \mathfrak{Y}) \in R^{p+1} \times R^{q+1}$ is a point of $S^p \times S^q$ if and only if

$$|\mathfrak{X}|^2 = \frac{p}{p + q} , \quad |\mathfrak{Y}|^2 = \frac{q}{p + q} .$$

2. Some new examples

As one observed in § 1, the classical examples of one-codimensional closed minimal submanifolds in S^m are exactly the real algebraic ones of degrees one and two. It was also observed that if one applies a theorem of [6] and some representation theory, then one obtains many more new examples, which are again homogeneous and hence real algebraic of degree greater than two. However, they are rather *scattered* dimensionally. For example, we get only two minimal cubic equations in five and eight variables respectively and only one minimal cone of degree six in 14 variables. It is very unlikely that they are the *only minimal algebraic cones*. In this section, we shall apply the classical invariance theory to get two new minimal cubics.

Lemma 2. *The differential operator*

$$M\cdot F = (\Delta F)\cdot|\nabla F|^2 - \nabla F\cdot HF\cdot\nabla F^t$$

is invariant under orthogonal linear substitutions.

Proof. Let $x_i = \sum_{j=0}^{m} a_{ij}\bar{x}_j$ be an orthogonal linear substitution, so that $A = (a_{ij})$ is an orthogonal matrix. Put

$$\bar{F}(\bar{x}) = F(x),\ \bar{\Delta} = \sum_{i=0}^{m} \partial^2/\partial\bar{x}_1$$

$$\bar{\nabla} = (\partial/\partial\bar{x}_0,\ \partial/\partial\bar{x}_1,\ \cdots,\ \partial/\partial\bar{x}_m) \text{ and } \bar{H} = (\partial^2/\partial\bar{x}_i\partial\bar{x}_j)\ .$$

Then

$$\bar{H}\bar{F} = \left(\frac{\partial^2\bar{F}}{\partial\bar{x}_i\partial\bar{x}_j}\right) = \left(\sum_{l,k=0}^{m}\frac{\partial^2 F}{\partial x_l\partial x_k}\cdot\frac{\partial x_l}{\partial\bar{x}_i}\cdot\frac{\partial x_k}{\partial\bar{x}_j}\right) = A^{-1}\cdot HF\cdot A$$

$$\bar{\nabla}\cdot\bar{F} = \left(\frac{\partial\bar{F}}{\partial\bar{x}_i}\right) = \left(\sum_{l=0}^{m}\frac{\partial F}{\partial x_l}\cdot\frac{\partial x_l}{\partial x_i}\right) = (\nabla F)\cdot A\ .$$

Hence $\bar{\nabla}\cdot\bar{F} = \text{trace}\ \bar{H}\cdot\bar{F} = \text{trace}\ H\cdot F = \Delta\cdot F$, and

$$\begin{aligned}\bar{M}\cdot\bar{F} &= (\bar{\Delta}\bar{F})\cdot|\bar{\nabla}\bar{F}|^2 - \bar{\nabla}\bar{F}\cdot\bar{H}\bar{F}\cdot\bar{\nabla}\bar{F}^t\\ &= (\Delta F)|\nabla F|^2 - \nabla F\cdot A\cdot(A^{-1}\cdot HF\cdot A)A^{-1}\nabla F^t = M\cdot F\ .\end{aligned}$$

Corollary. *In particular, if F is an invariant polynomial with respect to a given group G of orthogonal transformations, then so also is $M\cdot F$.*

Example 1. Let $\mathfrak{S}_4' = \boldsymbol{R}^9$ be the vector space of 4×4 real symmetric matrices with trace zero. Then 0(4) operates naturally on it. Let X be an arbitrary element in $\mathfrak{S}_4'$, and the characteristic polynomial of X be

$$\begin{aligned}\chi(X) = \det(X-\lambda I) &= \lambda^4 - \text{tr}(X)\lambda^3 + \beta_2(X)\lambda^2 - \beta_3(X)\lambda + \det(X)\\ &= \lambda^4 + \beta_2(X)\lambda^2 - \beta_3(X)\lambda + \det(X)\ .\end{aligned}$$

It is well known that $\beta_2(X)$, $\beta_3(X)$ and $\det(X)$ form a complete set of basic invariants with respect to the natural 0(4) action; notice that their degrees are 2, 3 and 4 respectively.

We claim that $\beta_3(X) = 0$ gives a new minimal cubic cone of 8 dimension in $\mathfrak{G}'_4 = \boldsymbol{R}^9$.

Proof. By Lemma 2, $M \cdot \beta_3(X)$ is also invariant, and hence $M \cdot \beta_3(X)$ can be expressed integrally in terms of the basic invariants $\beta_2(X)$, $\beta_3(X)$ and $\beta_4(X) = \det(X)$. Since the degree of $M \cdot \beta_3(X)$ is 5, the only possible such expression is

$$M \cdot \beta_3(X) = k \cdot \beta_2(X) \cdot \beta_3(X) \equiv 0 \quad (\text{mod. } \beta_3(X)) .$$

Hence $\beta_3(X)$ is a minimal cubic in 9 variables according to observation 3 in § 1.

Example 2. Let $\mathfrak{A}_3$ be the Lie algebra of $SU(4)$ with $SU(4)$ operating naturally as conjugations. $\mathfrak{A}_3$ may be represented as the set of 4×4 skew Hermitian matrices with trace zero and is a real vector space of dimension 15. Again, let $X \in \mathfrak{A}_3$ be an arbitrary element, and the characteristic polynomial be

$$\chi(X) = \det(X - \lambda I) = \lambda^4 + \gamma_2(X)\lambda^2 - \gamma_3(X)\lambda + \det(X) .$$

Then $\gamma_2(X)$, $\gamma_3(X)$ and $\gamma_4(X) = \det(X)$ are the set of basic invariants. The same reason will show that $\gamma_3(X) = 0$ gives a new minimal cubic in 15 variables.

The above two examples show that *the concept of real algebraic closed minimal hypersurfaces in S^m is, indeed, broader than that of homogeneous minimal hypersurfaces in S^m.* Although the procedure for getting the above two new examples is of quite special nature, there is no reason to believe that the two cubics in the examples are "isolated" minimal. It is then quite natural to ask the following problems of various depths.

Problem 1. How does one classify irreducible minimal cubic forms in m variables, $m \geq 4$, with respect to the natural action of $O(m)$? Or we may ask a weaker question, namely, whether there are always irreducible minimal cubic forms in m variables for all $m \geq 4$.

Problem 2. For a given dimension m, $m \geq 4$, are there irreducible homogeneous polynomials in m real variables of *arbitrary high degree*, which give minimal cones of codimension one in $\boldsymbol{R}^m$? Or, if the degree is bounded, how does one express the bound in terms of m?

Problem 3. Are there any closed minimal submanifolds of codimension one in S^m which are *not algebraic*? Or, if possible, show that *every closed minimal submanifold of codimension one in S^m is algebraic.*

Due to the fact that the differential operator M is very difficult to handle and the understanding of forms of degree higher than two is rather poor, the above problems seem to be far beyond our reach. We shall try instead a somewhat modest approach to some problems related to the ones stated above, which may help us to understand the difficulties involved.

3. Normal forms of homogeneous polynomials under orthogonal substitutions

It is well known that every quadratic form in n variables can be transformed into a "normal form" of sum of squares by suitable orthogonal linear substitution. As we can see in observation 4 of § 1, such a simple normal form for quadratics was very helpful for classifying minimal quadratic cones. One will naturally ask whether similar "normal forms" with respect to orthogonal substitutions exist for homogeneous polynomials of higher degree. To be precise, let us define what we mean by "normal forms."

Definition. Let $\mathfrak{H}_n^r$ be the vector space of homogeneous polynomials in n real variables of degree r. Then $O(n)$ acts naturally via linear substitutions on $\mathfrak{H}_n^r$, and the action is orthogonal with respect to the natural inner product on $\mathfrak{H}_n^r$. A linear subspace $\mathfrak{N}$ of $\mathfrak{H}_n^r$ is called a space of normal forms if $\mathfrak{N}$ is a *linear subspace* of least possible dimension which intersects every orbit of the above $O(n)$ action. Since $\dim O(n) = n(n-1)/2$, it is not difficult to see that $\dim \mathfrak{N} \geq \left(\dim \mathfrak{H}_n^r - \frac{n}{2}(n-1)\right)$.

Proposition 1. *Let $A \in \mathfrak{H}_n^r$ be a point on orbit $G(A)$ of maximal dimension. Then the linear subspace $\mathfrak{N}(A)$ of all normal vectors of $G(A)$ at A is a space of normal forms in $\mathfrak{H}_n^r$.*

Proof. We need to show that $\mathfrak{N}(A)$ meets every other orbit $G(B)$, $B \in \mathfrak{H}_n^r$. Since $G(A)$ and $G(B)$ are compact and disjoint, there exist $A' \in G(A)$ and $B' \in G(B)$ such that the segment $\overline{A'B'}$ realizes the shortest distance between $G(A)$ and $G(B)$. Surely, $\overline{A'B'}$ is normal to both $G(A)$ and $G(B)$. But, on the other hand, there exists a suitable element $g \in O(n)$ which is an orthogonal linear transformation of $\mathfrak{H}_n^r$ such that $g \cdot A' = A$. Hence, we see clearly that $g \cdot B' \in \mathfrak{N}(A) \cap G(B)$.

In the particular case where $r = 2$, the principal isotropy subgroups are the z_2-maximal tori (H) of $O(n)$. It is not difficult to see that $\mathfrak{N}(A) = F(H)$ for a suitable z_2-maximal torus $H \subseteq O(n)$. Since all z_2-maximal tori of $O(n)$ are conjugate, we see that the various subspaces of normal forms are also conjugate. Such phenomena cease to happen for $r \geq 3$ since the principal isotropy subgroups of the natural $O(n)$ action reduce to the trivial one, $\{e\}$, for $r \geq 3$. Hence, for $r \geq 3$, there are no "canonical" normal forms, and $\mathfrak{N}(A)$ depends essentially on the choice of A.

As an example to illustrate the *algebraic* meaning of the above proposition, we may take $A = \sum_{i=1}^{m} x_i^r$ for $r \geq 3$. Let

$$B = \sum \beta_{i_1,\dots,i_r} x_{i_1} x_{i_2} \cdots x_{i_r}$$

where $\beta_{i_1,\dots,i_r}$ are symmetric with respect to all indices. Then a direct computation will show that $B \in \mathfrak{N}(A)$ if and only if

$$\beta_{i,j,\cdots,j} = \beta_{j,i,\cdots,i} \quad \text{for all } i < j .$$

Notice that the above $n(n-1)/2$ conditions are linearly independent and hence the codimension of $\mathfrak{N}(A)$ is $n(n-1)/2$.

Remarks. (i) In view of a theorem of Tarski [7] parallel arithmetric statements are provable for every real closed field. To produce a direct algebraic proof of such a result for abstract real closed fields seems to be quite difficult.

(ii) Partly due to the lack of "canonical" normal forms for $r < 2$ and partly due to the rapid rate of increase of the dimension of $\mathfrak{H}_n^r$ with respect to r, the little help obtained from the normal forms is not enough to solve the problem of classifying minimal algebraic cones of higher degrees. For example, it is very difficult to solve even the following very special equation: $F(x) = 0$, where $F(x)$ is an irreducible cubic form in n variables such that

$$(\Delta F)\cdot|\nabla F|^2 - \nabla F\cdot HF\cdot\nabla F^t = \pm(x_1^2 + \cdots + x_n^2)\cdot F .$$

Since the above equation is invariant with respect to the orthogonal linear substitutions, we may assume that F is given in some kind of "normal form" which amounts to reduce the number of indeterminant coefficients by $n(n-1)/2$. A systematic attempt to solve the above equation will involve the job of solving over-determined simultaneous algebraic equations of many variables. So far, we have only four non-trivial solutions (cf. §§ 1, 2), but there is no reason why there should be no others.

4. A theorem on the geometry of closed minimal submanifolds of compact homogeneous spaces

Theorem. *Let $M^m = G/H$ be a compact connected manifold with homogeneous Riemannian metric, and N_1, N_2 two minimal submanifolds of codimension one in M^m. Then either $N_1 \cap N_2 \neq \phi$ or N_1, N_2 are geodesically parallel.*

In order to prove the above theorem, we need the following

Lemma 3. *Let $M^m = G/H$ be a compact connected manifold with homogeneous Riemannian metric, and x_1, x_2 two arbitrary points on M^m. Then there exists a one-parameter subgroup of isometries $A(t) \subseteq G$ such that*

$$A(1)\cdot x_1 = x_2,\ \delta(y, A(t)\cdot y) \leq \delta(x_1, A(t)\cdot x_1) \leq \delta(x_1, x_2)$$

for all $y \in G/H$ and $0 \leq t \leq 1$, where δ is the usual distance function.

Proof. Since M^m is connected, we may assume that G is also compact connected. The homogeneous Riemannian metric on G/H may be obtained as follows. We first impose on G a Riemannian metric such that it is invariant with respect to both right and left translations. It is well known that the cosets gH are then totally geodesic submanifolds and every geodesic curve may be considered as a part of a coset of certain one-parameter subgroups of G. It, then, naturally induces a Riemannian metric on the coset space $M^m = G/H$ such that $\delta(x, y) =$ the shortest distance between the two cosets x, y in the metric of G for all $x, y \in G/H$.

Without loss of generality, we may assume $x_1 = \{e\} = \{H\}$ and $x_2 = aH$. Since both H and aH are compact in G, there exists a geodesic arc $\overline{ha'}$ which realizes the shortest distance $\delta(x_1, x_2)$. If we translate $\overline{ha'}$ by a right translation $\rho_{h^{-1}}$, then the length of the geodesic arc $\overline{e(a'h^{-1})} = \delta(x_1, x_2)$. Let X be the element in the Lie algebra $\mathfrak{G}$ of G such that the geodesic arc $\overline{e(a'h^{-1})}$ is given by $\{\text{Exp } tX;\ 0 \leq t \leq 1\}$. We claim that $A(t) = \text{Exp } tX \subseteq G$ has all the desired properties. Let $y = g \cdot H$ be any point in G/H. Then we have

$$\begin{aligned}\delta(y, A(t)\cdot y) &= \delta(g\cdot H, A(t)\cdot gH)\\ &= \delta(H, g^{-1}A(t)gH) \leq \delta(e, g^{-1}A(t)g)\\ &= \delta(e, A(t)) = \delta(H, A(t)\cdot H)\\ &= \delta(x_1, A(t)\cdot x_1)\,.\end{aligned}$$

Proof of the theorem. Suppose $N_1 \cap N_2 = \phi$. Then $\delta(N_1, N_2) > 0$. Since N_1, N_2 are compact, there exist $x_1 \in N_1$ and $x_2 \in N_2$ such that $\delta(x_1, x_2) = \delta(N_1, N_2)$. By Lemma 3, there exists a one-parameter subgroup of isometries $A(t)$ such that

$$A(t)\cdot N_1 \cap N_2 = \phi \quad \text{for all} \quad 0 \leq t < 1$$

and $N_1' = A(1)N_1$ *touches* N_2 at x_2. Let ξ_1 be the common normal vector of N_1' and N_2 at x_2. Then we may choose suitable local coordinates $(u_1, u_2, \cdots, u_m)$ around x_2 and express the hypersurfaces N_1' and N_2 by

$$u_1 = f(u_2, \cdots, u_m),\ u_1 = g(u_2, \cdots, u_m)$$

respectively. Since N_1', N_2 are minimal, f, g satisfy the usual second order elliptic equation, and hence (f-g) satisfies another second order equation which is also elliptic. Since N_1' lies completely on one side of N_2, (f-g) has a local extremal at origin. It then follows from a theorem of E. Hopf (see, for instance, [8, p. 26]) that $f\text{-}g \equiv 0$ locally. By the analyticity of N_1', N_2 we see either that $N_1' = A(1)\cdot N_1 = N_2$, or that N_1 and N_2 are geodesically parallel.

References

[1] F. J. Almgren, Jr., *The theory of varifolds—A variational calculus in the large for the* k*-dimensional area integrand.*

[2] ——, *Some interior regularity theorems for minimal surfaces and an extension of Bernstein's theorem,* Ann. of Math. **84** (1966) 277–292.
[3] L. Eisenhart, *Riemannian geometry*, Princeton University Press, Princeton, 1949.
[4] H. Federer, *Some theorems on integral currents,* Trans. Amer. Math. Soc. **117** (1965) 43–67.
[5] T. Frankel, *On the fundamental group of a compact minimal submanifold,* Ann. of Math. **83** (1966) 68–73.
[6] W. Y. Hsiang, *On the compact homogeneous minimal submanifolds,* Proc. Nat. Acad. Sci. U.S.A. **56** (1966) 5–6.
[7] A. Tarski, *A decision method for elementary algebra and geometry,* Lithoprinted, University of California, Berkeley and Los Angeles, 1951, 63 pp.
[8] K. Yano & S. Bochner, *Curvature and Betti numbers,* Ann. of Math. Studies, Princeton University Press, Princeton, 1953.

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